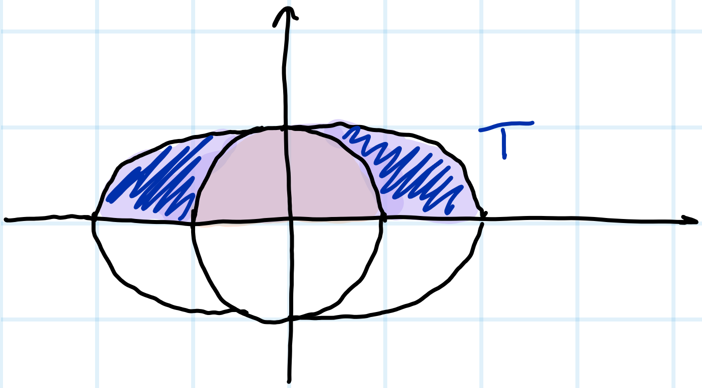
19/05/23 : Tutorato Analisi II

Esercizio

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in T, \ 0 \le z \le y(x^2+y^2)\}$$



$$T = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x^2+y^2 \ge 1 \\ x^2+4y^2 \le 4 \\ y \ge 0 \end{array}\right\}$$

$$|D| = \iiint_D dx\,dy\,dz = \iint_T \underbrace{\left(\int_0^{y(x^2+y^2)} 1 \cdot dz\right)}_{y(x^2+y^2) = f(x,y)} dx\,dy$$

$$T = \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x^2+4y^2 \le 4, \ y \ge 0\}}_{E} \setminus \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1, \ y \ge 0\}}_{C}$$

$$\iint_T f(x,y)\,dx\,dy = \iint_E f(x,y)\,dx\,dy - \iint_C f(x,y)\,dx\,dy$$

Dobbiamo parametrizzare in modo conveniente E e C.

C)
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \begin{array}{l} 0 \le \rho < 1 \\ 0 \le \theta \le \pi \end{array}$$

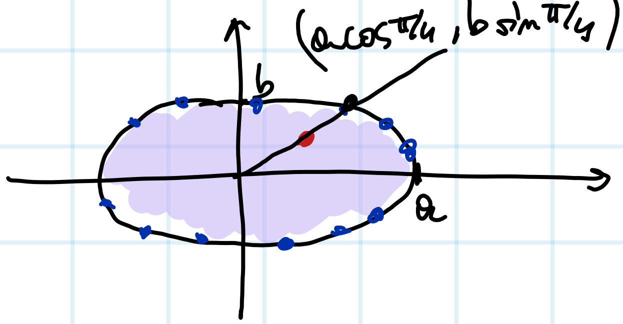
$$\int_0^{\pi}\int_0^1 f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

E) Parametrizzazione di una ellisse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = a\cos\theta \qquad y = b\sin\theta \implies \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2}{a^2}\cos^2\theta + \frac{b^2}{b^2}\sin^2\theta =$$

$$= \cos^2\theta + \sin^2\theta = 1$$



I punti "interni" sono parametrizzati da:

$$\begin{cases} x = \rho\, a\cos\theta \\ y = \rho\, b\sin\theta \end{cases} \qquad \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \end{array}$$

$$x = \rho\cos\theta \qquad \boxed{0 \le \rho \le a}$$

$$y = \rho'\sin\theta \qquad 0 \le \rho' \le b$$

$$0 \le \boxed{\frac{b}{a}\rho} \le a \cdot \frac{b}{a} = b$$

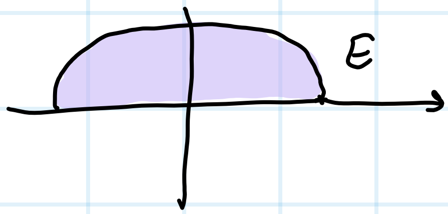
$$\begin{cases} x = \rho \cos\theta \\ y = \frac{b}{a}\rho \sin\theta \end{cases} \qquad \begin{matrix} 0 \le \rho \le a \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$\rho^2 \le a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \rho^2 \frac{\cos^2\theta}{a^2} + \rho^2 \frac{b^2}{a^2}\frac{\sin^2\theta}{b^2} \le$$

$$\le \frac{\cancel{a^2}}{\cancel{a^2}}\cos^2\theta + \frac{\cancel{a^2}}{\cancel{a^2}}\frac{\cancel{b^2}}{\cancel{b^2}}\sin^2\theta = 1$$

Sono entrambe parametrizzazioni della regione di piano data da $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$.



$$(\rho, \theta) \xmapsto{F} (\underbrace{a\rho\cos\theta}_{F_1}, \underbrace{b\rho\sin\theta}_{F_2})$$

$$\frac{\partial F_1}{\partial \rho} = a\cos\theta \qquad \frac{\partial F_1}{\partial \theta} = -a\rho\sin\theta$$

$$\frac{\partial F_2}{\partial \rho} = b\sin\theta \qquad \frac{\partial F_2}{\partial \theta} = b\rho\cos\theta$$

$$|\det J| = \left| \det \begin{pmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{pmatrix} \right| =$$

$$= |ab\rho\cos^2\theta + ab\rho\sin^2\theta| =$$

$$= ab\rho\,|\cos^2\theta + \sin^2\theta| =$$

$$= ab\rho$$

Tornando all'esercizio: $a=2$, $b=1$

$$\iint_E f(x,y)\,dx\,dy = \int_0^{\pi}\int_0^1 f(2\rho\cos\theta, \rho\sin\theta)\,\underbrace{\square}_{|\det J|}\,d\rho\,d\theta =$$

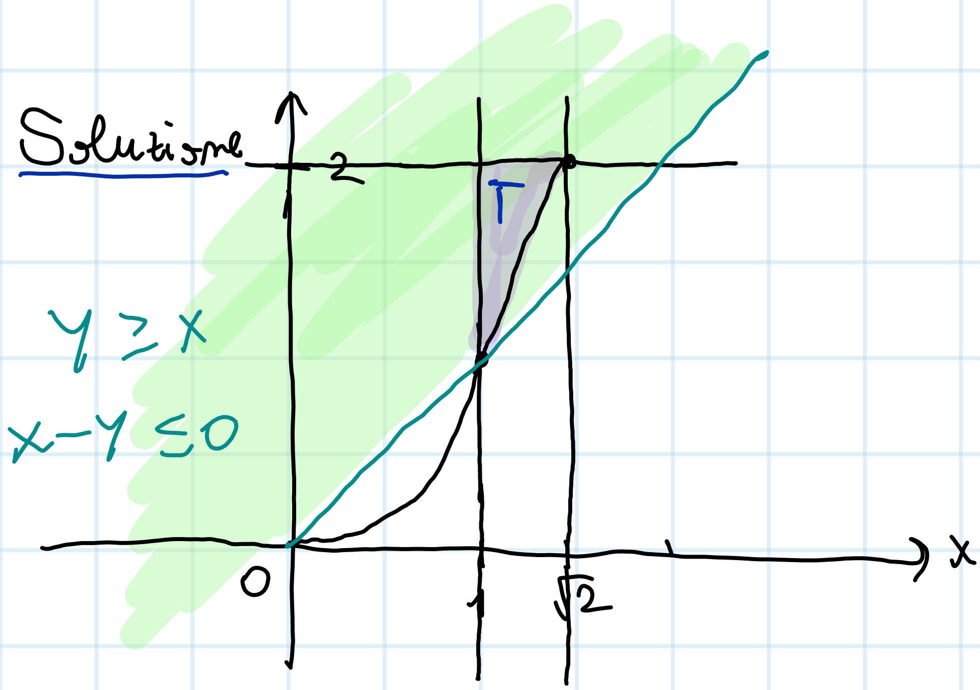
$$= \int_0^{\pi}\int_0^1 f(2\rho\cos\theta, \rho\sin\theta)\,2\rho\,d\rho\,d\theta$$

Esercizio $\iiint_D (x-y)z \, dx\,dy\,dz$

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in T, \ 0 \le z \le xy\}$$

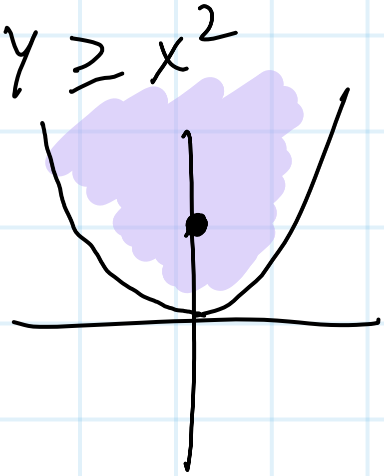
$$T = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le \sqrt{2}, \ x^2 \le y \le 2\}$$

Soluzione



$y=1$ $x=0$

$y \ge x^2$



$\underline{0 \le z} \le xy \Rightarrow z \ge 0$

in T abbiamo $y \ge x$

$f(x,y,z) = \underbrace{(x-y)}_{\le 0} \underbrace{z}_{\ge 0} \le 0$

in D

$$\iiint_D \overbrace{(x-y)z}^{f(x,y,z)} \, dx\,dy\,dz =$$

$$= \int_1^{\sqrt{2}} \int_{x^2}^{2} \int_0^{xy} (x-y)z \, dz \, dy \, dx =$$

$$= \int_1^{\sqrt{2}} \int_{x^2}^{2} (x-y) \left. \frac{z^2}{2} \right|_0^{xy} dy\, dx =$$

$$= \int_1^{\sqrt{2}} \int_{x^2}^{2} (x-y) \frac{x^2 y^2}{2}\, dy\, dx =$$

$$= \frac{1}{2} \int_1^{\sqrt{2}} \int_{x^2}^{2} (x^3 y^2 - x^2 y^3)\, dy\, dx =$$

$$= \frac{1}{2} \int_1^{\sqrt{2}} \left. \left(x^3 \frac{y^3}{3} - x^2 \frac{y^4}{4} \right) \right|_{x^2}^{2} dx =$$

$$= \frac{1}{2} \int_1^{\sqrt{2}} \left(\frac{8}{3} x^3 - 4x^2 - \frac{x^9}{3} + \frac{x^{10}}{4} \right) dx =$$

$$= \frac{1}{2} \left. \left(\frac{8}{3} \frac{x^4}{4} - 4 \frac{x^3}{3} - \frac{x^{10}}{30} + \frac{x^{11}}{44} \right) \right|_1^{\sqrt{2}} =$$

$= \ldots\ldots$

Esercizio 3

$$\iiint_S dx\,dy\,dz$$

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in T,\ g_1 \le z \le g_2\}$$

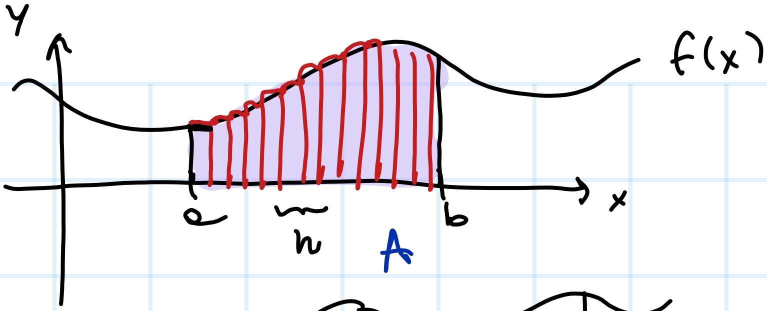
$$g_1(x,y) = x^2 + y^2 - 2 \quad , \quad g_2(x,y) = 4 - x - y$$

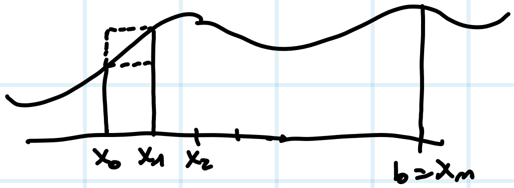
$$T = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$$

Soluzione (parziale)

$$\iiint_S dx\,dy\,dz = \iint_T \int_{g_1(x,y)}^{g_2(x,y)} dz\,dx\,dy =$$

$$= \int_0^{2\pi} \int_0^1 \int_{g_1(\rho\cos\theta,\rho\sin\theta)}^{g_2(\rho\cos\theta,\rho\sin\theta)} dz\ \rho\,d\rho\,d\theta \ .$$





intervalli "equispaziati"

h

n rettangoli

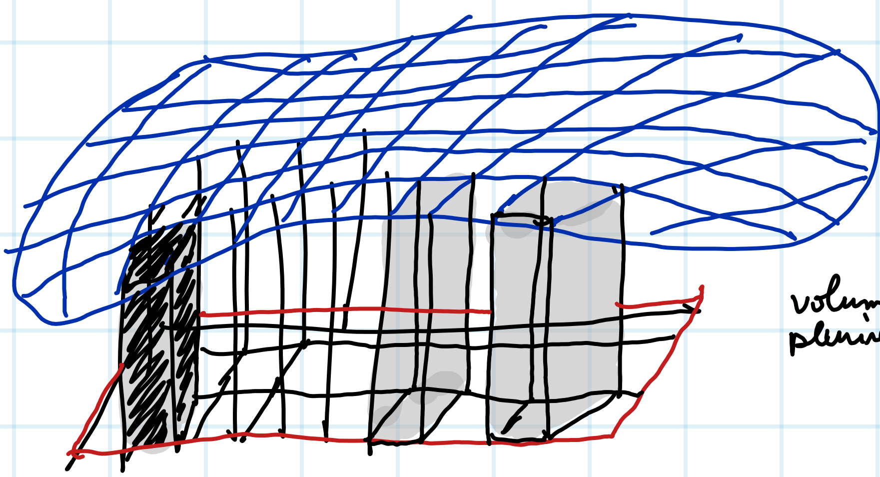
$$h = \frac{b-a}{n}$$

idea intuitiva:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \overbrace{\underbrace{h}_{\underbrace{x_i - x_{i-1}}_{\Delta x}} \cdot f(x_i)}^{\text{area rettangolo}} =$$

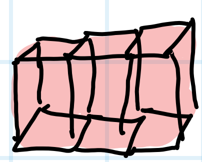
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x =$$

$$= \int_a^b f(x)\,dx$$



"plurirettangoli"

volume plurirettangolo

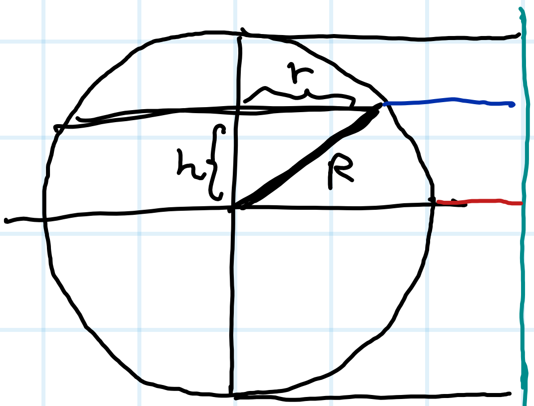


Altro modo: per strati

INTERPRETAZIONE INTUITIVA dell'INTEGRAZIONE PER STRATI:

VOLUME della SFERA di raggio R

ALTEZZA	AREA SEZIONE
R	0
h	$\pi(R^2-h^2)$
0	πR^2
$-R$	0



$$r^2 = R^2 - h^2$$

$$r = \sqrt{R^2-h^2}$$

V = volume sfera

"sommo" tutte le aree delle sezioni

$$V = \int_{-R}^{R} \underbrace{\text{sezione}(h)}_{\text{area sezione all'altezza } h}\, dh = \int_{-R}^{R} \pi(R^2-h^2)\, dh =$$

$$= \pi \int_{-R}^{R} R^2\, dh - \pi \int_{-R}^{R} h^2\, dh =$$

$$= \pi R^2 (\underbrace{R-(-R)}_{2R}) - \pi \left.\frac{h^3}{3}\right|_{-R}^{R} =$$

$$= 2\pi R^3 - \frac{\pi}{3}\left(\underbrace{R^3 - (-R)^3}_{2R^3}\right) =$$

$$= 2\pi R^3 - \frac{2}{3}\pi R^3 =$$

$$= \frac{4}{3}\pi R^3 \, .$$